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Wave jumps and caustics in the propagation of finite-amplitude water waves

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Nonlinear effects on the refraction of water waves are discussed. The existence of conjugate solutions for wave fields, one set of which corresponds to the 'anomalous' refraction solutions of Peregrine & Ryrie (1983) provides the stimulus for consideration of jumps in wave-field properties. Just such a jump is described by Yue & Mei (1980) for a case where near-linear waves are reflected with small deflection by a rigid wall.

Wave jumps between conjugate solutions appear to be possible for finite-amplitude wavetrains. These are examined and the structure of wave jumps in the near-linear, small-deflection case is elucidated. They have a structure directly analogous to that of an undular bore on shallow water with surface tension ('hydraulic analogy'). Numerical results of Yue & Mei (1980) provide valuable guidance and confirmation.

Linear waves are reflected at a caustic, and can be described with Airy functions. Although equivalent weakly nonlinear solutions exist, the results from reflection by a wall and from use of the hydraulic analogy show that, unlike the caustics of linear theory, nonlinear caustics should not be considered in isolation. Caustic cusps, or wave focusing, must be considered, unless bed topography has discontinuities. A qualitative discussion of focusing based on the behaviour of unsteady waves in the hydraulic analogy shows that wave jumps can be expected. The relationship to linear theory is also put in perspective. Nonlinearity causes the linear rays to split into two sets of characteristics. The splitting of a ray focus leads to two wave jumps.

Consideration of the case of a semi-infinite beach shows that anomalous refraction is most unlikely to occur because there is an offshore influence of the beach on the wave field which changes the incident wave conditions to prevent anomalous refraction.

1. Introduction

The refraction of finite-amplitude water waves has been studied in a series of papers using 'numerically exact' solutions for plane periodic water waves (Peregrine & Thomas 1979; Stiassnie & Peregrine 1980; Peregrine 1981; Ryrie & Peregrine 1982). These all describe examples where variables may vary in one space direction only, so that the governing equations for the averaged motion (Stiassnie & Peregrine 1979) can be integrated. Many of the solutions presented appear to be perfectly satisfactory, satisfying the assumptions that wave properties change slowly, except for the steepest waves where the maxima of the waves' integral properties lead to singularities which are sensibly interpreted as being an indication of wave breaking (for a discussion of this assumption and exceptions for long waves see Stiassnie & Peregrine 1980).

Other solutions are less satisfactory and have singularities at wave steepnesses well below the maximum, and a second solution branch is found. In the earlier works this always occurs in the neighbourhood of a caustic of linear theory, and the physical existence of the second solutions appeared doubtful; especially since no account is taken of waves reflected from the caustic region. However, in Ryrie & Peregrine (1982) the second solution appears to be appropriate in an apparently well-defined problem and has behaviour which differs qualitatively from that of linear theory. This 'anomalous' refraction is discussed in Peregrine & Ryrie (1983), where it is shown to be similar to the pairs of solutions near caustics and the properties of these 'conjugate' solutions are presented. Here the discussion is taken further and the problem that gave anomalous refraction is shown to be ill-posed.

This paper has two major topics: jumps in wave properties between conjugate solutions are described and their implications for wave propagation near linear caustics is investigated. A particularly helpful example is given by Yue & Mei (1980, hereinafter referred to as YM). In that paper the existence of wave jumps is recognized for waves incident almost parallel to a wedge of small angle. The solutions computed by YM involve two approximations: (i) that the waves are only weakly nonlinear and (ii) that the waves are only deflected through a small angle. However, diffraction effects are retained, as in a parabolic approximation, and are important.

Wave jumps for waves of finite amplitude and for finite changes of wave direction must always be oblique to the wave-propagation direction and can only exist for a modest range of angles. Details of amplitude and jump direction are given for the case of deep water waves, and for solitary waves, incident on an inclined half plane (i.e. the problem analysed by YM). The solitary-wave case is the Mach reflection discussed by Miles (1977*b*).

The structure of the wave jumps that appear in YM's solutions is analysed in §3. It is noted that the modulations that occur are similar to an undular bore and solutions of the governing nonlinear Schrödinger (NLS) equation support this view. In particular, it is shown that the NLS equation can be cast into a form analogous to Boussinesq's equations for shallow water waves. This is called a 'hydraulic analogy' in the subsequent discussion, in order to distinguish between the three levels of waves that are considered: (i) the original wavetrain and its development, (ii) the modulated envelope of those waves and (iii) the long-wave solutions of the Boussinesq equations which are used to describe the qualitative properties of solutions of the NLS equation.

The interpretation of YM's solutions is relevant to the reflection of waves which is expected to occur in the neighbourhood of a linear caustic. A local nonlinear solution exists in which a Painlevé transcendent replaces the Airy function of linear wave theory. However, YM's solution for a reflecting plane indicates that these solutions are inappropriate and it is necessary to consider the initiation of a caustic. Caustics normally originate in pairs from cusps, which represent an imperfect focus of waves.

Use of the hydraulic analogy makes it a simple matter to discuss the qualitative structure of the wave field near a focus. The nonlinear terms contribute to a defocusing (the self-focusing NLS equation has been studied more in the past, see Whitham 1974, §16.3). For a sufficiently strong focus a wave jump can be expected to occur each side of the focus and no identifiable caustics arise. This implies that the wave field can differ considerably from that of linear wave theory. The general behaviour is similar to that of non-dispersive waves, where both theoretical and experimental results are known for sound waves (Cramer & Seebass 1978; Sturtevant & Kulkarny 1976; Fridman 1982). Together with the analogy between gasdynamics and shallow-water flows, this suggests that the qualitative picture may be unchanged throughout a surf zone on a beach.

In §6 anomalous refraction, which is analogous to subcritical hydraulic flow, is discussed. Some sort of 'control' (such as occurs at a weir or lake on a stream) is needed before it can be realized. The implication of this for waves approaching a beach at a very oblique angle is that the wave conditions offshore are changed to reduce the steepness of waves approaching the beach.

The concluding discussion assesses the problems involved in reproducing these various effects in a laboratory, or observing them in nature. Indications are given of the further work necessary to use these results in practical wave prediction, and for determining their range of applicability. It is noted that the wave field must vary slowly since experiments in which waves focus in a region with dimensions of the order one wavelength show a strong transfer of energy to the second harmonic.

2. Wave jumps

The first papers to discuss jumps of water-wave properties concerned the undular bore which occurs in shallow water. Benjamin & Lighthill (1954) show that, within the cnoidal-wave approximation which is appropriate to the undular-bore problem, the only one-dimensional transition in wave properties for an irrotational flow which conserves mass, momentum and energy is between a uniform supercritical flow and a solitary wave. The similarity between the leading wave of an undular bore and a solitary wave has been remarked on by several authors.

The possibility of wave jumps is discussed in Whitham's (1965) first paper on the averaging of nonlinear wavetrains (see Whitham 1974, §15.4). (The term 'jump' is used here rather than 'shock', as used by Whitham, since the latter term describes a well-defined phenomenon in gasdynamics, whereas the former term is commonly used in many contexts for a sharp change in the value of physical and/or mathematical quantities.) In agreement with Benjamin & Lighthill (1954), Whitham finds that jumps for unidirectional waves are not possible unless some constraint is relaxed.

However, the nonlinear averaging method leads to solutions which develop sharp gradients of wave properties suggesting that jumps might allow a uniformly valid solution. Howe (1968) discusses in detail how one such solution might be extended by inclusion of an oblique wave jump. Energy and momentum conservation are used to locate Howe's jump, but a phase discontinuity exists across it. Consideration of that example in the light of the results developed in this paper suggests that, if diffraction effects are neglected, two jumps must arise from the region of focused wave energy and these would satisfy all the conservation equations. In practice, it would probably be difficult to discern a jump in most experiments of that type.

Ostrovskii (1968) gives a relatively full description of wave jumps for a one-dimensional system of modulation equations which is equivalent to a generalization of the nonlinear Schrödinger equation. Jump structure is analysed for dissipative cases. Although §3 parallels Ostrovskii (1968), a conservative system is being considered and we proceed to a further interpretation of wave jumps.

The evaluation of properties of very steep water waves by Schwartz (1974), Longuet-Higgins (1975) and Cokelet (1977) shows that energy, mass and momentum properties of steep waves are not single-valued functions of wave properties. The consequent possible jumps are recognized in Longuet-Higgins & Fenton's (1974) discussion of high solitary waves which show the same behaviour. Chen & Saffman (1980) show that there is a bifurcation of the wavetrain solution at the maximum phase velocity. The jumps and transitions of wave properties in this region are not discussed here since in practical applications waves of this steepness will rapidly break. See Longuet-Higgins's (1978*a*, *b*) analysis of deep-water wave stability and the reviews by Schwartz & Fenton (1982) and Peregrine (1983*a*). The conclusions of Benjamin & Lighthill (1954) are confirmed by results from more accurate wave solutions.

Other water-wave jumps have been recognized. Chin (1979) describes wave jumps, but uses unrealistic values in detailed examples so that no firm conclusion can be reached from his study. Jumps in the propagation direction of solitary waves are analysed independently by Reutov (1976), Ostrovskii & Shrira (1976) and Miles (1977*c*). These are treated in a similar manner to such jumps on shock waves; see 'shock–shocks' in Whitham (1974, §8.6). Miles's (1977*a*, *b*) study of intersecting and 'resonantly interacting' solitary waves provides the detail of such jumps (Mach reflection); this shows a weak reflected wave which is consistent with the jump structure discussed in the next two sections.

Yue & Mei (1980) give numerical solutions for near-linear waves incident on a wedge of small angle. They recognize that their solutions show wave jumps at a small angle to the wave direction. The position of the jump and the change in wave amplitude are shown to be consistent with a jump model. Their approximation is restricted to small changes in the direction of wave propagation. The conjugate solutions discovered for finite-amplitude waves, discussed by Peregrine & Ryrie (1983) allow an extension to finite angles and amplitudes.

In its simplest form a wave jump consists of a plane across which wave properties change between two uniform values. For such a jump to be possible all appropriate quantities must be conserved across the jump. Discussion here is in terms of water waves and steady wave fields, but extensions to other types of waves and to steadily moving jumps is possible.

There are two 'kinematic' constraints on a jump. The incident waves force the wave field on the other side (except for any example with zero energy flux across the jump). Thus the wave frequency and the component of wavenumber along the jump are conserved. Whitham (1965, 1974) suggests that these requirements might be relaxed, probably since only one-dimensional waves are considered, but except for the possible excitation of waves with frequencies and/or wavenumber components which are some integer multiple of the incident values, the constraints of geometric and time periodicity seem paramount. The physical constraints are that mass, momentum and energy are conserved; but these might be relaxed if there is some reflection (see §4).

For water waves, there are six conserved quantities:

(i) ω, wave frequency relative to the jump;

(ii) k_2, wavenumber component along the jump;

(iii) q_1, total mass flow across the jump;

(iv), (v) S_{11}, S_{12}, total flux of momentum across the jump;

(vi) F_1, total flux of energy across the jump;

where Ox_1 is taken normal to the jump and Ox_2 is along the jump.

Seven quantities are needed to define conditions on each side of the jump. They are

ω, wave frequency;

$(k_1, k_2) = \boldsymbol{k}$, wavenumber;

$(U_1, U_2) = \boldsymbol{U}$, mean velocity;

D, mean depth;

a, wave amplitude.

If conditions are known on one side of the jump then the six equations arising from the conserved quantities supplemented by the dispersion equation are sufficient, in

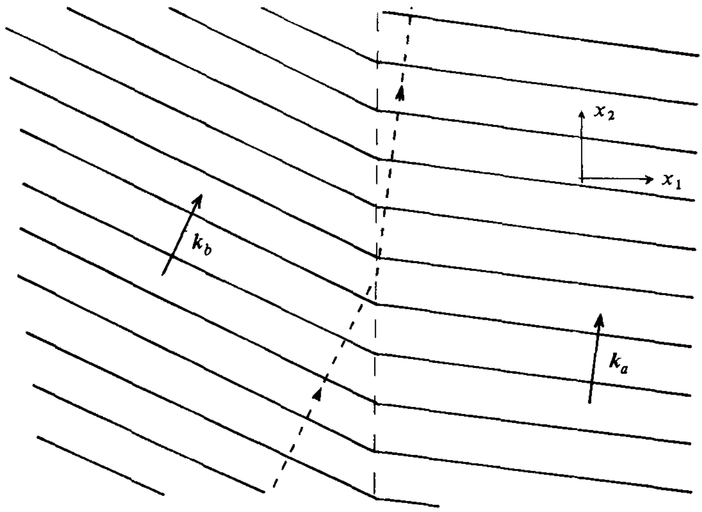

FIGURE 1. Sketch of wave jump, continuous lines represent wave crests. No representation of the structure of the jump is included.

principle, to determine conditions on the other side of the jump. These equations, or their equivalent, are solved in refraction problems where the propagation medium and wave properties only vary slowly in the Ox_1 direction, though energy conservation is usually replaced by wave-action conservation (e.g. see Ryrie & Peregrine 1982).

The case of deep water waves is a limiting case where the velocity and mean depth are specified, so the mass and momentum conservation equations cannot be used. To be consistent, a large-scale two-dimensional mean flow should be incorporated in some larger-scale analysis (e.g. see McIntyre 1981). Further study is required, but it is unlikely that simpler analysis which omits such a mean flow will have significant errors if wave action is substituted for wave energy.

It is not enough to postulate that solutions of the conservation equations exist. It must be shown that they have two solutions: one for each side of the jump. Peregrine & Ryrie (1983) present such conjugate solutions for zero mean flow. The results show that stationary jumps with no mean flow are limited to a certain range of obliquely incident waves by the fact that water waves have a maximum steepness.

A sketch of a wave jump, showing wave crests and lines parallel to $\boldsymbol{k}$, is given in figure 1. The arrows indicate a progression from side b (for before) to side a (for after) with waves on side a propagating more nearly parallel to the jump.

The analysis leading to conjugate solutions gives no indication of the sense of the jump. An equivalent to entropy in a shock wave or energy dissipation in a hydraulic jump would be useful. However, the study of jump structure in §3 shows that for near-linear waves with a small angle of deviation, the propagation properties of modulations clearly show that waves incident on a jump are lower than those after the jump, in agreement with the sketch. It is reasonable to suppose that the same result is true when those approximations are relaxed, although no method is available to calculate jump structure.

A simplified representation of the wave field described by YM is given in figure 2. Since there are no wave-jump solutions with wave propagation normal to a jump, this configuration appears to be the simplest realistic possibility for jump creation. The waves are incident from the left on a half-plane or wedge face inclined at an angle

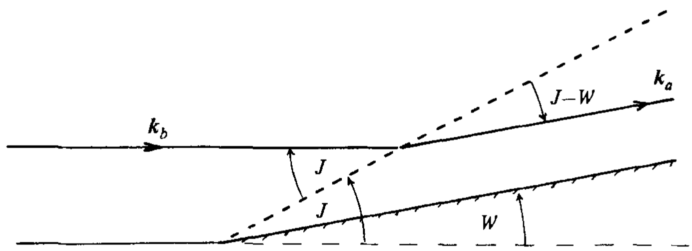

Figure 2. Schematic diagram of a wave field, with wavenumber $\boldsymbol{k}_b$, incident on a rigid half-plane, or wedge at an angle W, forming a wave jump at an angle J.

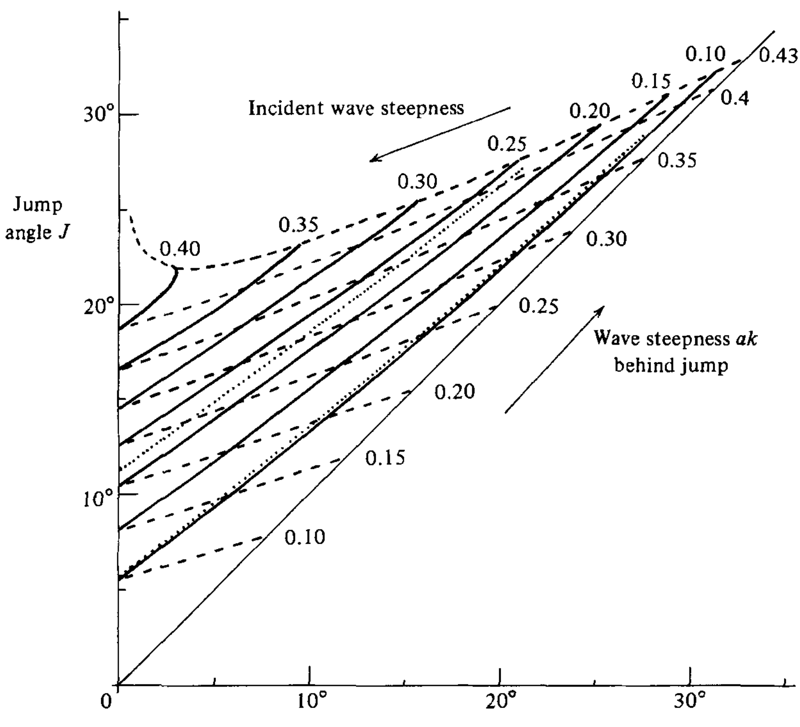

Figure 3. Solutions for wave steepness before and after deep water waves meet a jump of angle J caused by a wedge of angle W. Numerically accurate periodic wave solutions have been used. The dotted lines show the relation between J and W for initial steepnesses of 0.1 and 0.2 from the approximation of Yue & Mei (1980).

W to the incident wave direction. The edge of the half-plane induces the formation of a jump at an angle J to the incident wave direction. The waves behind the jump must travel parallel to the plane if they are to be represented by a single wavetrain.

For a simplified deep-water wave analysis, conservation of wave action and the along-jump component of wavenumber can be used to find the jump angle J and wedge angle W after specifying the wave steepness before and after the jump. Results are shown in figure 3. YM's analysis corresponds to the lower left-hand corner of the diagram, two curves obtained by using their approximate relation (4.21) are included.

For waves in finite depth of water, the configuration of figure 2 is not as simple as it looks. The mass-flow condition normal to the jump together with the

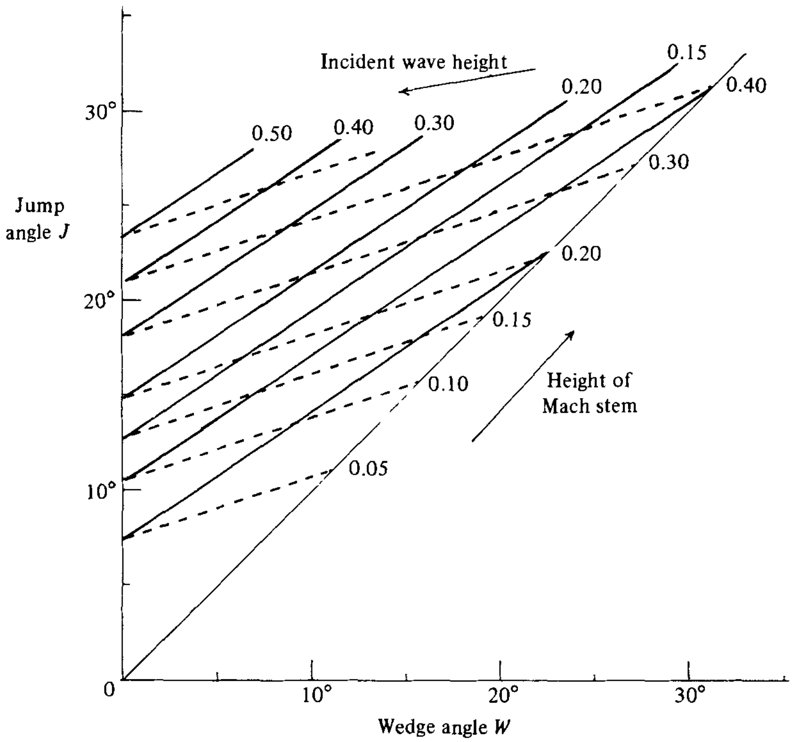

FIGURE 4. Results from Miles's (1977*b*) near-linear analysis of Mach-stem reflection of solitary waves interpreted as if due to a wave jump at a wedge. The wave heights are in units of the undisturbed depth.

impermeability of the wedge mean that the current and wavenumber must both have the same orientation after the jump. This is an extra constraint on the jump and an incident mean flow is needed to completely satisfy the jump conditions. However, since the magnitude of the flow involved is the same as the mass flow associated with the wavetrains, differences from the simple picture of figure 2 are likely to be slight. For example, in the small-deviation, near-linear approximation of YM, wave-induced currents are implicitly included. The relatively large width associated with the jump structure causes such details to have little practical importance.

The limit of very long waves is the solitary wave. The configuration corresponding to figure 2 and to YM's solutions is that of Mach reflexion (Miles 1977*b*). The Mach stem can be identified with the region between the jump and the wedge. Figure 4 gives results corresponding to those of figure 3, derived from Miles's results. Funakoshi (1980) has confirmed with numerical computations that for finite angles the small-angle results of Miles are a good approximation to solutions of Boussinesq's equations. No finite-amplitude solutions are available.

In YM's solutions the wave crests along the wedge are not perpendicular to the line of the wedge. This is inconsistent with an impervious boundary. YM show that the discrepancy is $O(\epsilon^3)$, and hence one may expect that if the approximation is carried out to this higher order the discrepancy is likely to be eliminated.

3. Structure of wave jumps

The structure of wave jumps can be found by improving the accuracy of the solution to resolve shorter lengthscales. For the case of jumps at a small angle to the incident and transmitted waves this is done in YM for near-linear waves. YM derive a nonlinear Schrödinger (NLS) equation for these water waves. The same equation can be derived in other ways and for other wave systems (see e.g. Peregrine & Smith 1979, §§6 and 7; Grimshaw 1981).

The waves are assumed to be steady with the primary propagation direction being Ox. The nonlinear effects on the velocity of wave propagation and the rate of variation of wave properties in the transverse, Oy, direction are assumed to be of the same order of magnitude, i.e. $\partial^2/\partial y^2 = k^2 O(a^2k^2)$, with a longer modulation in the x-direction, $\partial/\partial x = O(a^2k^3)$.

The NLS equation can be written in dimensionless form as

$$2\mathrm{i}A_T + A_{YY} - K|A|^2 A = 0, \tag{3.1}$$

where $T = k_0 x$, $Y = k_0 y$, and k_0 is the wavenumber of the basic wavetrain. The constant K, which is positive,† depends on the relative water depth and on exactly what measure of the amplitude is denoted by A, which is a complex function of (Y, T). The variable T is used here because the modulations are compared with an analogous time-varying case. Equation (3.1) can be derived directly by simplification of the Davey & Stewartson (1974) equations: put terms with ξ-derivatives negligible compared with the others in their equation (2.15).

For gravity water waves K is positive for all depths and equation (3.1) is that form of the NLS equation which has stable solutions of uniform amplitude:

$$A = A_0 \mathrm{e}^{-\frac{1}{2}\mathrm{i}K|A_0|^2 T}. \tag{3.2}$$

Ostrovskii (1968) demonstrated that wave jumps are possible for (3.1) between uniform solutions and that they have an oscillatory character. We interpret these oscillations in two ways.

The numerical solutions of YM clearly show the structure of their wave jumps, see YM figures 3, 4, 6 and 7. Transverse sections of the wave envelope in YM figure 4 show that the jump looks like an undular bore propagating away from the wedge. Alternatively the detailed view of water surface elevation in YM figure 7 emphasizes the similarity with Mach reflection of a solitary wave and the region of transition between the two uniform wave regions can then be identified as a region with waves which have partially reflected off the higher waves (the Mach-stem region). In this section we confirm the undular bore analogy, and in §4 consider implications for the reflections of nonlinear waves.

That wave jumps have modulations preceding them is consistent with the property of the linearized modulations that group velocity is faster than phase velocity. There is no upper limit to the group velocity; however, the modulation equations are not valid for rapid modulations. In undular bores the longest wave, which adjoins a uniform level, grows steadily and becomes very similar to a solitary wave. Here similar behaviour can be seen for the largest modulation.

† The sign of K is crucial to the discussion of wave jumps and to the rest of the paper. For general near-linear wave systems, its sign corresponds to the sign of H/G_{ll} in the notation of Peregrine & Smith (1979, p. 347). For fully nonlinear systems similar behaviour might be found in cases where long perturbations have real characteristics.

The analogy with an undular bore can be carried further by rewriting equation (3.1) in terms of real variables defined as follows:

$$A = a\,e^{i\psi}, \quad h = a^2, \quad v = \frac{\partial \psi}{\partial Y}. \tag{3.3}$$

The resulting equations are

$$h_T + (hv)_Y = 0, \tag{3.4}$$

$$v_T + vv_Y + \tfrac{1}{2}Kh_Y = \frac{1}{4}\left(\frac{h_{YY}}{h} - \frac{h_Y^2}{h^2}\right)_Y. \tag{3.5}$$

These or similar equations have often been likened to those of gasdynamics, but a comparison with shallow-water equations is more instructive. First note their interpretation as modulation equations. Equation (3.4) is a 'small-angle' approximation to the linearized wave-action equation, and (3.5) is a differentiated version of the dispersion equation with $\frac{1}{2}Kh_Y$ representing the near-linear terms. The right-hand side is an approximation to diffraction effects on dispersion which may be more readily recognized when it is in the form $(a_{YY}/2a)_Y$.

In the shallow-water analogy, $h(Y,T)$ corresponds to water depth, $v(Y,T)$ to velocity, (3.4) to the mass-conservation equation, and (3.5) to the momentum equation with $\frac{1}{2}K$ corresponding to the acceleration due to gravity and an unusual right-hand side. Clearly if variation with Y is relatively slight the diffraction term of (3.5) may be neglected and the finite-amplitude shallow-water equations result.† If wave jumps are considered in the simplified, discontinuous, form as in §2, there is a discrepancy between bores in shallow-water flow and the jumps obtained in this context. In the case of bores it is mass and momentum which are conserved through a jump, whereas here it is wave action and frequency which correspond to mass and velocity respectively. However, this does not affect the analogy given below.

Perturbations about a uniform wavetrain $h = 1$ (the initial amplitude is included in the parameter K by YM) given by $h = 1+\eta$ lead to a first approximation to the diffraction term of $\frac{1}{4}\eta_{YYY}$. This is a common form for the dispersive term in Boussinesq's equations for weakly dispersive shallow-water waves (e.g. see equation (13.9) of Whitham 1974). This weakly nonlinear dispersive approximation will be termed a 'hydraulic' analogy, to distinguish its long water-wave solutions from both the water waves of the original problem and their wave-like modulations. As in the usual derivations of Boussinesq's equations, the nonlinear terms on the left-hand side are retained, even with the linearized form of the diffraction term as long as the modulations satisfy

$$\frac{\partial^2}{\partial Y^2} = O(\eta). \tag{3.6}$$

The sign of the dispersive term in the hydraulic analogy is such that surface tension must be dominant, e.g. see Korteweg & de Vries (1895), and the coefficients of the analogous Boussinesq equations agree exactly for depth D, surface tension T and density ρ if

$$\frac{T}{\rho g D^2} - \frac{1}{3} = \frac{1}{2K}. \tag{3.7}$$

For example, if $K = 1$, the analogy is with long waves on water of 3 mm depth.

Thus, unless modulations are $O(1)$, solutions for the corresponding 'hydraulic'

† Note that negative values for K correspond to water lying under a rigid boundary with a free surface below. Solutions for $K < 0$ have a completely different character which is not investigated here (but see Peregrine 1983*b*).

problem may be used. If initial conditions correspond to a smooth modulation between two uniform conditions propagating along wave crests it is equivalent to a long hydraulic wave. The resulting steepening (which is analysed for the NLS equation by Ostrovskii 1968) eventually leads to the rate of modulation becoming great enough to 'activate' the diffraction term. The corresponding development of an undular bore has been studied for the hydraulic case, numerically by Peregrine (1966) and Fornberg & Whitham (1978) and analytically by Gurevich & Pitaevskii (1974), who use Whitham's method of averaging nonlinear waves (the same approach could be used here). Although these papers describe the gravity-dominated hydraulic waves, all the solutions are readily transformed to describe the surface-tension-dominated case.

The initial wave of the undular bore grows until it is like a solitary wave of about twice the initial change of level. Similar behaviour can be expected for the final modulation in a wave jump, except that for large changes in modulation it is limited by the maximum modulation deepening to zero. If this latter limit is attained the upper level of the jump will stop propagating, since the velocity of the limiting solitary modulation is zero. YMs results do not extend far enough to show this, but their figure 7 does show that the steeper wavetrain spreads more slowly as x increases.

Despite the fact that the hydraulic analogy is not precise for large modulations of waves with diffractive effects, the qualitative comparison holds good. Both these modulations and surface-tension-dominated water waves have velocities which approach zero as the trough of the modulation or wave approaches zero. Similarly for large peaks the 'restraining' effects of the diffraction term and surface tension are qualitatively similar.

4. The wave field near caustics

From considerations of a single nonlinear wavetrain Peregrine & Smith (1979) show that caustics fall into two classes. We only consider the R-type caustics here; S-type caustics have $K < 0$.

In the neighbourhood of a caustic position, the single-wavetrain approximation has a singularity where two solution branches merge. For example see Peregrine & Thomas (1979), Peregrine (1981), Ryrie & Peregrine (1982) and Peregrine & Ryrie (1983). These two branches are conjugate in the sense that a wave jump can occur between them. However, that particular wave jump is most unlikely, as is now explained.

According to linear theory waves reflect at a caustic; in a ray representation rays touch caustics. Thus for weakly nonlinear waves it is reasonable to expect the linear caustic line to act as a reflector and to correspond to the plane reflector in YM's example. In the case of the reflecting plane no steady state is reached, the region of steep waves continues to grow in extent, and these waves have no component of wave-action flux towards the reflector. The wave field near a linear caustic is likely to have the same character. There are solutions for waves near a caustic with no component of wave-action flux towards the caustic, these waves also have 'anomalous' properties relative to the caustic line and hence can be conjugate to the incident waves across a wave jump (see e.g. figure 2 of Peregrine & Thomas 1979 (delete the last 5 words of the caption) and figures 6 and 7 of Ryrie & Peregrine 1982). However, the wave jump would have to be at an angle to the caustic line since only the linear zero-amplitude incident wave is conjugate to the zero-wave-action-flux component solution.

There is a local steady solution for near-linear waves in the vicinity of a linear caustic, it involves a Painlévé transcendent in place of the Airy function Ai(x) of linear theory. Peregrine & Smith (1979) introduce the Painlévé transcendent in this context, Rosales (1978) and Miles (1980) give details of its properties, and it has a region, between asymptotic-oscillatory and exponentially decaying regions, which corresponds to waves propagating parallel to the caustic. The conclusion to be drawn from YM's solutions is that this steady solution is only obtained asymptotically some distance along the caustic if at all. Kirby & Dalrymple (1983) present a computed solution which shows the initial development of a wave jump near a linear caustic.

The wave field along a caustic thus varies with distance from the initial point of the caustic.

In a wave field without discontinuities of the refracting medium or of the initial wave fronts, caustics begin in pairs at cusps of caustics, which represent imperfect focuses of the waves. Hence it is necessary to consider focusing.

5. Focusing of nonlinear waves

The focusing of waves satisfying the NLS equation has been studied in the context of nonlinear optics (e.g. Akhmanov, Sukhorukov & Khokhlov 1966). The effect of the nonlinear term is such that there is either 'self-focusing' or 'defocusing' according to whether K is negative or positive respectively in the NLS equation (3.1). This difference depending on the sign of K corresponds also to the S and R types of caustics respectively (Peregrine & Smith 1979). Most attention has been given to the self-focusing case; we make use of the hydraulic analogy to describe the defocusing case appropriate to gravity water waves.

The hydraulic flow corresponding to a wave field directed towards a focus has water converging on a stagnation point. The effect of gravity (nonlinear dispersion for the waves) is to diminish the height of the hydraulic flow at the stagnation point and cause it to flow away again; this is the defocusing effect. The resulting outgoing flow has the form of a long wave of elevation which has a tendency to steepen and then form undulations. The speed of steepening and the resulting occurrence of an undular wave jump depends upon the wave steepness, the degree of focusing and the scale of the focusing region. Figure 3 of Stamnes *et al.* (1983) shows good examples.

It is instructive to compare the ray approximation to linear theory with the hydraulic analogy where the diffraction terms are ignored. They are equivalent approximations since ray theory gives the position of caustics and their cusps, but fails in their neighbourhood, and similarly the finite-amplitude shallow-water equations give the position of the wave jumps but not their structure.

In the hydraulic analogy, particle paths correspond to lines parallel to the wavenumber vector $\boldsymbol{k}$ (rays in linear theory). Figures 5 (a, b) give sketches of the linear rays at a caustic cusp and the particle paths in a nonlinear hydraulic flow with a representation of wave jumps spreading away from the focusing region. The figure gives an impression that waves from the right, say, of a focus form a wave jump on the right side of the focal region instead of a caustic on the left side. Figure 5 (b) is, however, misleading, in the hydraulic flow information propagates along characteristics, and it's these which are equivalent to the rays of linear theory, not the wavenumber direction. As figure 5 (c) shows, each set of characteristics forming a jump comes from the same side as the rays forming the corresponding caustic in the linear case. That is, the effect of nonlinearity is to split the linear characteristics and also to split a cusp of caustics into a separate cusp of the envelope of each family of

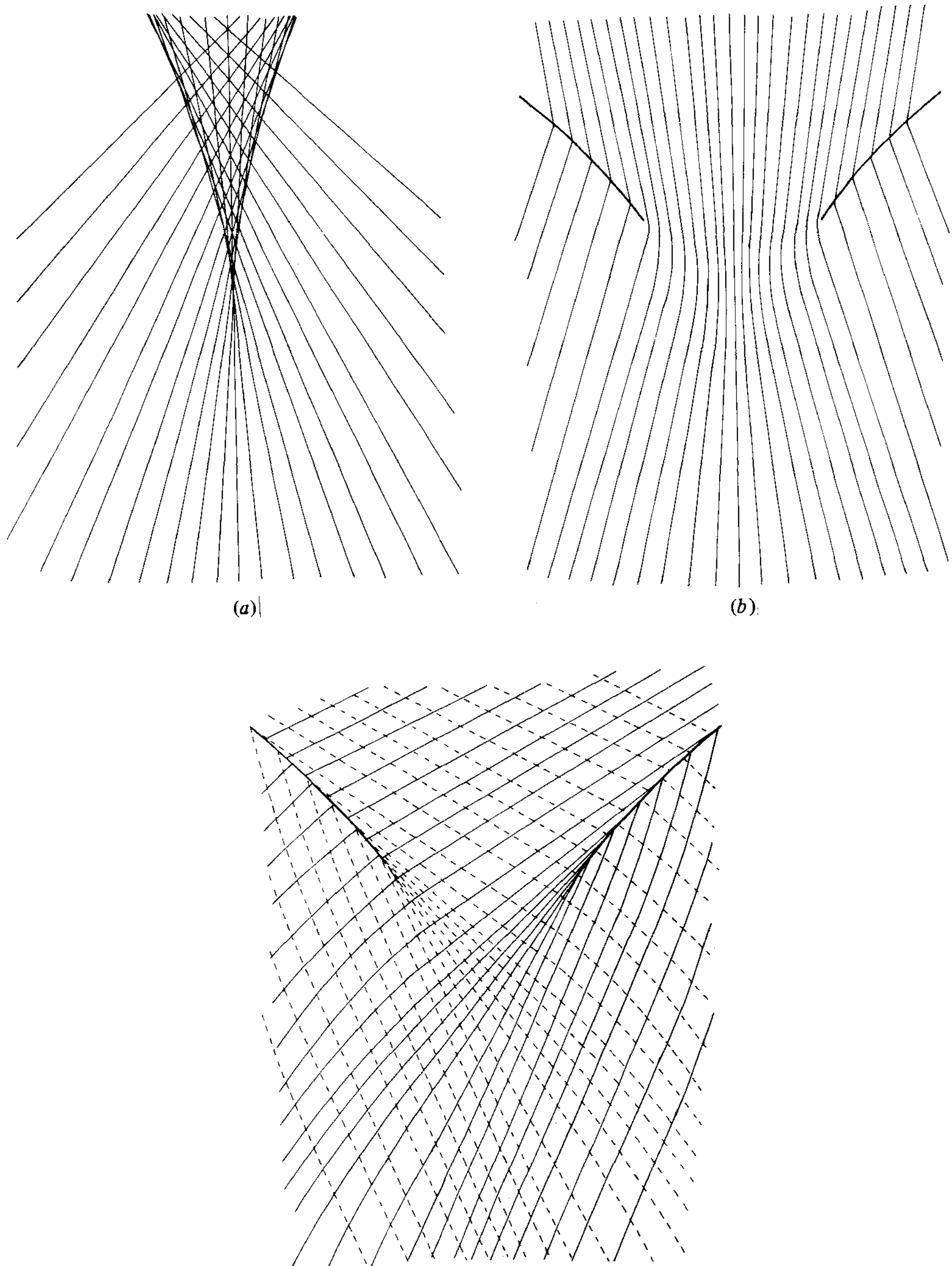

FIGURE 5. (*a*) Rays at a cusp of caustics, linear theory. (*b*) Sketch of the particle paths of the hydraulic analogy at a focus, jumps included with no structure. The lines correspond to lines parallel to $\boldsymbol{k}$, the wavenumber in the wave field. (*c*) Sketch of characteristics of the wave field, and its analogous hydraulic flow, in the case where diffraction terms, and jump structure are ignored.

characteristics. Once the jump has formed, its position is determined by both sets of characteristics.

The further development of wave jumps or caustics depends on a larger portion of the wave field and can vary substantially. This subsequent development of the wave field and clarification of parameters appropriate to near-linear focusing requires considerable further work.

6. Anomalous refraction

In a typical water-wave refraction problem the 'initial' wavetrain is specified, e.g. wave steepness, period and direction may be specified for waves incident on a beach. Normally this is a satisfactory procedure. However this is not satisfactory in the cases, such as waves incident almost parallel to a beach, where anomalous refraction occurs. The 'small-angle' near-linear approximation of the NLS equation is appropriate to such a case, and there is no loss of generality in considering the constant-depth case.

The analysis which leads to anomalous refraction includes an assumption of long modulations so that an approximation which ignores the diffractive terms on the right-hand side of (3.5) is also appropriate. The resulting, shallow-water equations have real characteristics, and it is clear from the general theory of hyperbolic equations that boundary conditions involving a complete specification of the wave-train are only properly posed when both sets of characteristics proceed into the region of integration. It is easily seen that anomalous refraction solutions occur only for ill-posed problems.

In the case of a straight beach full specification of the incident wavetrain is incorrect when characteristics propagate away from the beach. These characteristics propagate offshore and modify the wave field so that it arrives in the area of the beach with properties appropriate for normal refraction, in which all characteristics propagate towards the beach. This behaviour is illustrated in figure 6, where the beach starts at $x = 0$, the incident waves being 'guided' there by a vertical wall. In $x > 0$ there is a 'simple-wave' fan of offshore-propagating characteristics through which the waves are turned more nearly normal to the beach.

There is a direct analogy with sub- and supercritical flows in channels and the upstream influence of any obstacle in a subcritical flow. The anomalous refraction case corresponds to subcritical flow.

These results also apply to finite-amplitude waves. The problem of ill-posedness of nonlinear wave-propagation problems is discussed in Hayes (1973). Deep-water waves correspond to the case of Hayes' (1973) §4, and finite-water depth is introduced in §5. Taking the simpler case of §4, we are concerned here with steady wave modulations, which in Hayes's notation means that

$$v + \boldsymbol{n}\cdot\boldsymbol{c} = 0. \tag{6.1}$$

This is just the Doppler relation between the phase velocity v in the frame of reference moving with the basic group velocity $\boldsymbol{c}$ relative to the medium; $\boldsymbol{n}$ is a unit vector in the direction of the modulation wavenumber. In terms of the Hamiltonian $\mathscr{H}(A, \boldsymbol{k})$, where A is wave action, (6.1) is

$$\boldsymbol{n}\cdot\mathscr{H}_{Ak} = -(\mathscr{H}_{AA}\mathscr{H}_{kk}:\boldsymbol{nn})^{\frac{1}{2}}, \tag{6.2}$$

which is an equation for $\boldsymbol{n}$.

It may be seen from Hayes's equation (4.13) that this equation for $\boldsymbol{n}$ defines the critical direction, relative to $\boldsymbol{k}$, for the normal to a (mathematical) boundary. One

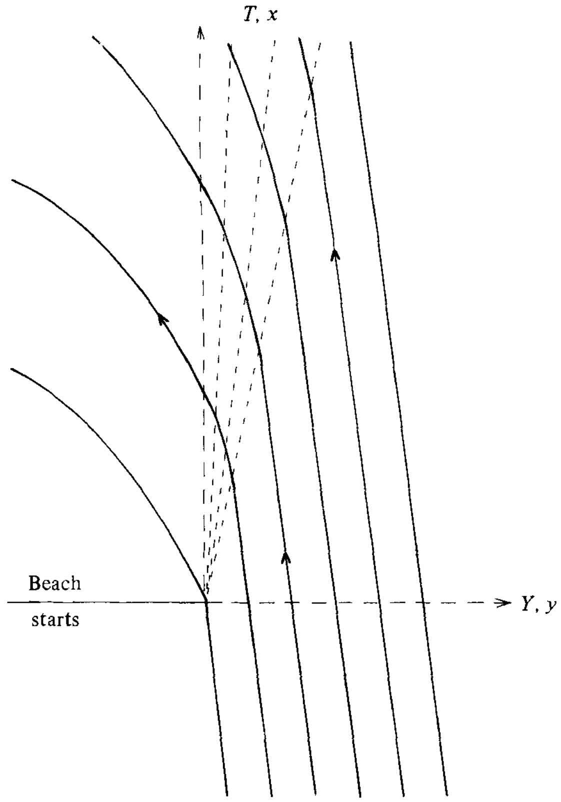

FIGURE 6. Representation of waves meeting a beach which starts at $x = 0$. The continuous lines are parallel to the wavenumber. The broken lines are the fan of characteristics in a simple wave.

side of the critical direction gives boundary lines on which the wave field may be specified and the solution problem is well-posed, and a boundary on the other side of the critical direction the problem is ill-posed.

Equation (6.2) is more directly related to the near-linear case described in connection with figure 6. The group velocity of modulations is given by Hayes (1973) as

$$\boldsymbol{c}+\boldsymbol{c}_1 = \mathscr{H}_{\boldsymbol{k}A}+\frac{\mathscr{H}_{AA}\mathscr{H}_{\boldsymbol{kk}}\cdot\boldsymbol{n}}{(\mathscr{H}_{AA}\mathscr{H}_{\boldsymbol{kk}}:\boldsymbol{nn})^{\frac{1}{2}}}. \tag{6.3}$$

Equation (6.2) corresponds to

$$\boldsymbol{n}\cdot(\boldsymbol{c}+\boldsymbol{c}_1) = 0 \tag{6.4}$$

The modulations propagate perpendicular to $\boldsymbol{n}$, along the lines of constant phase, and are thus directly analogous to the near-linear characteristics.

Further, (6.2) may be squared and rewritten as

$$\frac{\partial(\mathscr{H}_{\boldsymbol{k}}, \mathscr{H}_A)}{\partial(A, \boldsymbol{k}\cdot\boldsymbol{n})} = 0, \tag{6.5}$$

which corresponds to

$$\frac{\partial(\boldsymbol{B}, \omega)}{\partial(ak, \cos\theta)} = 0, \tag{6.6}$$

using the relations for wave-action flux $\boldsymbol{B}$ and frequency given by Hayes, and noting that A is a function of ak for given $\boldsymbol{k}$; the angle between $\boldsymbol{k}$ and $\boldsymbol{n}$ is $\frac{1}{2}\pi-\theta$. The dividing line in the (ak, θ)-plane between regular and conjugate solutions is given by

$$\frac{\partial(b_1, m_2)}{\partial(ak, \theta)} = 0, \tag{6.7}$$

in Peregrine & Ryrie (1983). In that restricted case it is straightforward to show the equivalence of (6.6) and (6.7).

For finite water depth there are two more characteristics with velocities close to that of long water waves, $(gh)^{\frac{1}{2}}$. This long-wave velocity is greater than other perturbation velocities unless the primary wavetrain is composed of long waves. Excepting this latter case, Hayes' (1973) analysis for these waves implies that the wave-field modulations are qualitatively similar to those of deep-water waves. However, due account must be taken of other boundaries in determining the distribution of mean depth and current.

Note that for both finite-depth and deep-water waves the restriction to steady wave fields implies that we exclude from consideration the unsteady modulations considered by Hayes and others, and known to be unstable in many cases.

7. Discussion

The entire development of this paper is theoretical, yet the predictions for reflection and refraction are such that the effects should be observable in experiments. Experiments on waves incident at a small angle to a plane are described by Berger & Kohlhase (1976). The experiments are compared with linear theory, and show reasonable agreement. It is possible to discern some lessening of amplitude at the wall and flattening of the crests perpendicular to the wall for the steeper waves as one would expect from YM's solutions. However, the oscillations in the measurements are of the order 25 % so that any deductions from their results are doubtful.

There are two major difficulties in confirming any of the present results with experiment. One is the problem of scale. Since we describe modulations which are long compared with the wavelength, hundreds of wavelengths are needed and this means a large physical scale is necessary to avoid significant dissipative effects. A comparison of Melville's (1980) experimental measurements of the Mach reflection of a solitary wave with Funakoshi's (1980) numerical modelling shows that Melville was limited in the scale of his experiments. Similarly, Whalin's (1972) experimental focusing of water waves by refraction led to a focal region which was of the same scale as the wavelength, in which energy was transferred to a free second harmonic.

The other problem is that the waves should be steady and periodic. Even if waves are generated without any free harmonics, they are vulnerable to instabilities as they propagate. The Benjamin–Feir instability is well known to affect waves in water depths with $kD > 1.36$, and the experiments of Su *et al.* (1982*a*, *b*) and theory of McLean (1982) show that even for $kD < 1.36$ instabilities can grow. However, by suitable choice of water depth and wave steepness it should be possible to avoid these. In this theoretical account all time dependence is omitted; it would be valuable to extend this work to describe the reflection and refraction of modulated incident waves.

The physical existence of the finite-amplitude wave jumps described in §2 and 3 has yet to be demonstrated. Experiments and observations are only available for solitary waves and the related area of gasdynamics. The nonlinear equations differ in these cases from those for waves on deep or moderate depths of water. The primary difference is that, in shallow water, wave speeds increase linearly with amplitude, whilst in deeper water the increase is with the square of the amplitude. However, as has already been noted by comparisons between wave jumps and Mach reflection, the qualitative effects are similar. Presumably these primarily depend on the increase of wave speed with amplitude.

The simpler type of approximation, which describes wave jumps but not their structure, has been developed independently for solitary waves by Reutov (1976), Ostrovskii & Shrira (1976), and Miles (1977*c*). Examples of the refraction of a solitary wave have been discussed by Kulikovskii & Reutov (1976, 1980).

In the analogous area of weakly nonlinear acoustics, experiments on focused waves described by Sturtevant & Kulkarny (1976) show features described here in §5. Corresponding to our wave jumps, 'shock–shocks' are observed. The clearest summary is in their figure 18 (their photographs include the complications of diffraction from the edge of their focusing reflector and the optical effects of heated gas). For theoretical aspects see Whitham (1974, chap. 8) and Cramer & Seebass (1978).

The qualitatively similar behaviour of solitary waves and acoustic waves encourages the hope that development of the approaches used here might even have some value through regions of beaches where waves are breaking. The bores that often develop once waves break are analogous to acoustic shock waves. Walker (1976) describes focusing refraction in an example where waves break, and his comparison with linear theory shows nonlinear defocusing.

The limits of the approximations have not been well defined, although rough estimates can be made. For example, in deep water one might expect $ak \approx 0.2$ to be a reasonable upper limit on steepness for the NLS equation. Very little information is available about the maximum reasonable modulation rate.

The verification of any refraction theory for natural waves on natural beaches is difficult since even in the best circumstances the incident waves are of varying amplitude. The effects of such variation have yet to be worked out. Of all the phenomena described here, the offshore influence for waves almost parallel to a beach is likely to be the least difficult to observe.

The major points of this paper are that (i) wave jumps may exist at a finite angle to finite-amplitude water waves; (ii) for near-linear waves with small angles of deviation wave jumps are similar in structure to undular bores and involve partial reflection of wave energy; (iii) the analogy between the NLS equation (3.1) and the flow of water a few millimetres deep allows a qualitative description of nonlinear refraction phenomena which includes a description of focusing.

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